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INTERVAL MATRIX SYSTEMS – FLOW INVARIANCE AND COMPONENTWISE ASYMPTOTIC STABILITY

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Abstract. Given an interval matrix system with discrete- or continuous-time dynamics, the flow-invariance of a rectangular set with arbitrary time-dependence is introduced as a concept of geometric nature. A theorem for its characterization provides a link between the vector function describing the time-dependence of the rectangular set and a constant matrix, which, in a certain sense, dominates the interval matrix. For rectangular sets with exponential time-dependence, this link becomes a system of algebraic inequalities. If there exist flow-invariant rectangular sets approaching the state space origin for an infinite time horizon, then the interval matrix system exhibits two special types of asymptotic stability (called componentwise asymptotic stability - for arbitrarily time-dependent rectangular sets, and componentwise exponential asymptotic stability - for exponentially time-dependent rectangular sets), whose equivalence is proved. A necessary and sufficient condition for componentwise asymptotic stability is derived, which adequately exploits Schur or Hurwitz stability of the constant matrix dominating the interval matrix.

1. Introduction

The concept of flow-invariant time-dependent sets has been exploited in several works for studying particular properties of the solutions of various types of differential equations and it is based on the pioneering research developed at the middle of the century by Nagumo [9], and Hukuhara [6]. Later on, in early seventies, significant contributions to the characterization of flow-invariance have been brought by Brezis [3], Crandall [5] and Martin [7]. Remarkable monographs on this field are due to Pavel [10], Motreanu and Pavel [8]. The use of rectangular sets has been proposed by Voicu [13], [14] for continuous-time linear constant systems, resulting in the definition

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and analysis of a special type of (exponential) asymptotic stability, namely the componentwise (exponential) asymptotic stability. An overview on the application of the flow-invariance method in control theory and design is given in [15], referring to the case of continuous-time nonlinear dynamical systems.

Our work aims to create a *unified framework* for addressing the flow-invariance of time-dependent rectangular sets with respect to *interval matrix systems* in both *discrete-* and *continuous-time* cases. The research has been motivated by the current interest manifested in model construction for using intervals instead of fixed coefficients, in order to cope with incomplete information and uncertainties. The approach also allows us to introduce and explore the componentwise (exponential) asymptotic stability of interval matrix systems, as well as to outline the connections with some recent results on Schur or Hurwitz stability of interval matrices, reported by Chen [4], Bauer and Premaratne [1], Sezer and Šiljak [11]. In this context, it is worth mentioning that the conditions formulated in [1], [11] only as *su cient* for the standard concept of asymptotic stability of interval matrix systems (defined in terms of any consistent norm), become *necessary* and *su cient* for the componentwise asymptotic stability. Thus, the meaning of the necessity part, which does not hold for the results in [1], [11], is fully explained by the supplementary constraints obeyed by the state-space trajectories when the interval matrix system is componentwise asymptotically stable.

The paper is structured according to the following plan that permits concomitant discussions on discrete- and continuous-time interval matrix systems. Section 2 introduces specific notation and presents two results necessary for handling matrix inequalities and for unifying the discrete- and continuous-time approaches. Section 3 studies the flow-invariance with respect to an interval matrix system in the general case of arbitrarily time-dependent rectangular sets, whereas the interest of Section 4 is restricted to the particular situation of exponential time-dependence. Section 5 and Section 6 are dealing with the componentwise asymptotic stability and, respectively, the componentwise exponential asymptotic stability of interval matrix systems; connections with the standard concept of asymptotic stability are amply discussed.

2. Preliminaries

The purpose of this section is to present the specific notations and algebraic background to be used in the remainder of the text.

Taking into consideration the particular nature of problems arisen by the flow-invariance method when operating with inequality-type constraints, we

consider two arbitrary matrices $\mathbf{P}$, $\mathbf{Q}$ of the same size, the entries of which are referred to as p_{ij} and q_{ij} respectively. The following componentwise (element–by–element) matrix inequalities are defined:

$$\begin{array}{l} \mathbf{P} < \mathbf{Q} \ \text{ means } \ \forall\, i,j : p_{ij} < q_{ij}, \\ \mathbf{P} \leq \mathbf{Q} \ \text{ means } \ \forall\, i,j : p_{ij} \leq q_{ij}. \end{array} \tag{2.1}$$

If $\mathbf{P}$ is a real matrix, the notation $|\mathbf{P}|$ will be used for a nonnegative matrix of the same size as $\mathbf{P}$, whose entries are the absolute values $|p_{ij}|$ of the entries p_{ij} of $\mathbf{P}$. Obviously the meaning of all these notations introduced above holds for the particular situation when handling vectors, instead of matrices.

Matrices that will later occur in our discussion on flow invariance are characterized by the nonnegativeness of their off-diagonal entries, inducing the dominant role of an eigenvalue with respect to the location of the whole spectrum, as stated below:

Lemma 2.1. *Let $\mathbf{P}$ be a matrix with nonnegative o -diagonal entries (i.e., an essentially nonnegative matrix) and denote by $\lambda_i(\mathbf{P})$, $i = 1, ..., n$, the eigenvalues of $\mathbf{P}$.*

(i) If the diagonal entries of $\mathbf{P}$ are nonnegative, then $\mathbf{P}$ has a real nonnegative eigenvalue (simple or multiple) denoted by $\lambda_{\max}(\mathbf{P})$, which fulfills the dominance condition:

$$|\lambda_i(\mathbf{P})| \leq \lambda_{\max}(\mathbf{P}), \quad i = 1, \ldots, n. \tag{2.2}$$

(ii) If the diagonal entries of $\mathbf{P}$ are arbitrary, then $\mathbf{P}$ has a real eigenvalue (simple or multiple), denoted by $\lambda_{\max}(\mathbf{P})$, which fulfills the dominance condition:

$$\Re e[\lambda_i(\mathbf{P})] \leq \lambda_{\max}(\mathbf{P}), \quad i = 1, \ldots, n. \tag{2.3}$$

Proof. (i) If the diagonal entries of $\mathbf{P}$ are nonnegative, then $\lambda_{\max}(\mathbf{P}) = \rho(\mathbf{P})$, where $\rho(\mathbf{P})$ denotes the spectral radius of $\mathbf{P}$ (cf. [2]).

(ii) If the diagonal entries of $\mathbf{P}$ are arbitrary, then it can be written as:

$$\mathbf{P} = \mathbf{Q} - s\mathbf{I}, \quad \mathbf{Q} \geq 0, \ s \geq 0, \tag{2.4}$$

and shows that the eigenvalues of $\mathbf{P}$ and $\mathbf{Q}$ are related by the following equalities:

$$\lambda_i(\mathbf{P}) = \lambda_i(\mathbf{Q}) - s, \quad i = 1, \ldots, n. \tag{2.5}$$

It results that $\lambda_{\max}(\mathbf{P}) = \rho(\mathbf{Q}) - s$, where $\rho(\mathbf{Q})$ denotes the spectral radius of the nonnegative matrix $\mathbf{Q}$. □

Further development of our work will need handling particular types of systems of inequalities, reason for which a necessary and sufficient condition ensuring the compatibility is derived below:

Lemma 2.2. *Let $\mathbf{P}$ be a matrix with nonnegative o -diagonal elements and denote by $\lambda_{\max}(\mathbf{P})$ its dominant eigenvalue in the sense formulated in Lemma 2.1. The system of inequalities:*

$$\mathbf{P}\mathbf{w} \leq p\mathbf{w}, \quad p \in \mathbb{R}, \quad \mathbf{w} \in \mathbb{R}^n, \quad \mathbf{w} > 0, \tag{2.6}$$

is compatible (has solutions) if and only if the following condition is met:

$$\lambda_{\max}(\mathbf{P}) \leq p\,. \tag{2.7}$$

Proof. We are first dealing with the situation when the diagonal entries of $\mathbf{P}$ are nonnegative. Let $\mathbf{w} = [w_1 \ \ldots \ w_n]^T \in \mathbb{R}^n$ be a positive vector and consider the diagonal matrix

$$\mathbf{D} = \operatorname{diag}\{w_1, \ldots, w_n\} \in \mathbb{R}^{n\times n}, \quad w_i > 0, \quad i = 1, \ldots, n. \tag{2.8}$$

If the system of inequalities (2.6) is expanded as

$$\frac{1}{w_i} \sum_{j=1}^{n} p_{ij} w_j \leq p, \quad i = 1, \ldots, n, \tag{2.9}$$

one can simply notice that (2.6) is equivalent to

$$\|\mathbf{D}^{-1}\mathbf{P}\mathbf{D}\|_\infty \leq p. \tag{2.10}$$

On the other hand, the following result is available for the nonnegative matrix $\mathbf{P}$ (cf. [12]):

$$\lambda_{\max}(\mathbf{P}) = \rho(\mathbf{P}) = \min_{\mathbf{D}} \|\mathbf{D}^{-1}\mathbf{P}\mathbf{D}\|_\infty, \tag{2.11}$$

where $\mathbf{D}$ is defined by (2.8) and has arbitrary positive diagonal entries. Thus, inequality (2.10) holds if and only if (2.7) is true.

Now we address the situation when the diagonal entries of $\mathbf{P}$ are arbitrary. Let us consider the system of inequalities

$$\mathbf{Q}\mathbf{w} \leq q\mathbf{w}, \quad q \in \mathbb{R}, \quad \mathbf{w} \in \mathbb{R}^n, \quad \mathbf{w} > 0, \tag{2.12}$$

where $\mathbf{Q}$ is the nonnegative matrix occurring in expression (2.4) used in the proof of Lemma 2.1. From the preceding step of the current proof, it results that system (2.12) is compatible if and only if $\lambda_{\max}(\mathbf{Q}) = = \rho(\mathbf{Q}) \leq q$. Hence the system of inequalities

$$(\mathbf{Q} - s\mathbf{I})\mathbf{w} \leq (q - s)\mathbf{w}, \quad q \in \mathbb{R}, \quad \mathbf{w} \in \mathbb{R}^n, \quad \mathbf{w} > 0, \tag{2.13}$$

is compatible if and only if $\lambda_{\max}(\mathbf{Q}) = \rho(\mathbf{Q}) \leq q$. On the other hand, taking into account (2.4), one can simply see that (2.13) is identical to (2.6) for $p = q - s$, and $\lambda_{\max}(\mathbf{P}) = \rho(\mathbf{Q}) - s$. In other words, system (2.6) is compatible if and only if $p = q - s \geq (\lambda_{\max}(\mathbf{P}) + s) - s = \lambda_{\max}(\mathbf{P})$. □

We are now in the possession of all the requested notations and algebraic results and, consequently, we can proceed to the exploration of flow invariance.

3. Flow-invariance of arbitrarily time-dependent rectangular sets with respect to an interval matrix system

In order to unify the exposition for *discrete-time* and *continuous-time interval matrix systems*, we are going to use an *operator* $(\ )'$ which, applied to a discrete-time function, supplies the one step ahead value of the function, and, applied to a continuous-time function, supplies the value of the derivative (if exists). To formalize this, we consider the vector function $\mathbf{x}(t)$, where $t \in \mathbb{T}$ denotes the independent variable (with discrete-time or continuous-time meaning), and define the *action of the operator* $(\ )'$ as:

$$\mathbf{x}'(t) = \begin{cases} \mathbf{x}(t+1), & t \in \mathbb{T} = \mathbb{Z}_+ \quad (\text{discrete} - \text{time case}), \\ \dot{\mathbf{x}}(t), & t \in \mathbb{T} = \mathbb{R}_+ \quad (\text{continuous} - \text{time case}). \end{cases} \tag{3.1}$$

Now, taking advantage of this notation, the free response of an interval matrix system (regardless of its discrete-time or continuous-time nature) can be described by the equation:

$$\mathbf{x}'(t) = \mathbf{A}^I \mathbf{x}(t); \quad \mathbf{x}(t_0) = \mathbf{x}_0; \quad t, t_0 \in \mathbb{T}, \quad t \geq t_0. \tag{3.2}$$

In (3.2) $\mathbf{A}^I$ denotes an *interval matrix* defined as a family of real matrices:

$$\mathbf{A}^I = \{\mathbf{A} \in \mathbb{R}^{n \times n} : \quad \mathbf{A}^- \leq \mathbf{A} \leq \mathbf{A}^+\}, \tag{3.3}$$

which is able to capture the uncertainties affecting the model construction for the studied dynamics.

Definition 3.1. The rectangular set with arbitrary time-dependence:

$$\mathcal{H}(t) = [-h_1(t),\, h_1(t)] \times \ldots \times [-h_n(t),\, h_n(t)], \quad h_i(t) > 0, \tag{3.4}$$

$i = 1, \ldots, n, \ t \in \mathbb{T}$, is *flow-invariant* with *respect* to interval matrix system (3.2) if for any $t_0 \in \mathbb{T}$ and any initial condition $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathcal{H}(t_0)$, the whole state trajectory $\mathbf{x}(t)$ corresponding to $\mathbf{x}(t_0)$ remains inside the set $\mathcal{H}(t)$, i.e.,

$$\forall\, t_0, \quad t \in \mathbb{T}, \quad t \geq t_0, \quad \forall\, \mathbf{x}(t_0) = \mathbf{x}_0 \in \mathcal{H}(t_0) : \ \mathbf{x}(t) \in \mathcal{H}(t). \tag{3.5}$$

This definition is given in terms of the state-space trajectories of interval matrix system (3.2), and our interest now focuses on the characterization of flow-invariance of $\mathcal{H}(t)$, (3.4), with respect to interval matrix system (3.2) by using information offered directly by the interval matrix $\mathbf{A}^I$ (3.3).

Theorem 3.1. *The rectangular set with arbitrary time-dependence* $\mathcal{H}(t)$, (3.4) *is flow-invariant with respect to interval matrix system* (3.2) *if and only if*

$$\forall\, t \in \mathbb{T}: \quad \overline{\mathbf{A}}\mathbf{h}(t) \leq \mathbf{h}'(t), \tag{3.6}$$

where matrix $\overline{\mathbf{A}}$ *is constructed from the interval matrix* $\mathbf{A}^I$ (3.3) *as follows*: (i) *for discrete-time case*:

$$\overline{a}_{ij} = \sup_{\mathbf{A}\in\mathbf{A}^I} \{|a_{ij}|\} = \max\{|a_{ij}^-|, |a_{ij}^+|\}, \quad i,j = 1,\ldots,n; \tag{3.7}$$

(ii) *for continuous-time case*:

$$\overline{a}_{ii} = \sup_{\mathbf{A}\in\mathbf{A}^I} \{a_{ii}\} = a_{ii}^+, \quad i = 1,\ldots,n, \tag{3.8-a}$$

$$\overline{a}_{ij} = \sup_{\mathbf{A}\in\mathbf{A}^I} \{|a_{ij}|\} = \max\{|a_{ij}^-|,\ |a_{ij}^+|\}, \quad i \neq j, \quad i,j = 1,\ldots,n. \tag{3.8-b}$$

Proof. (i) *discrete-time case*. The fulfillment of the flow-invariance condition stated in Definition 3.1 means that, for any $t \in \mathbb{Z}_+$, the inequalities given below hold

$$-h_i(t+1) \leq \sum_{j=1}^{n} a_{ij}x_j(t) \leq h_i(t+1), \quad i = 1,\ldots,n, \tag{3.9}$$

with

$$a_{ij}^- \leq a_{ij} \leq a_{ij}^+, \quad i,j = 1,\ldots,n, \tag{3.10}$$

$$-h_j(t) \leq x_j(t) \leq h_j(t), \quad j = 1,\ldots,n. \tag{3.11}$$

Applying constrains (3.10) and (3.11) to a_{ij} and $x_j(t)$ in (3.9), and, then using the notations introduced by (3.7), one can notice that (3.9) becomes equivalent to two systems of n simple inequalities:

$$-h_i(t+1) \leq \sum_{j=1}^{n} \overline{a}_{ij}(-h_j(t)), \quad i = 1,\ldots,n, \tag{3.12-a}$$

$$\sum_{j=1}^{n} \overline{a}_{ij}h_j(t) \leq h_i(t+1), \quad i = 1,\ldots,n. \tag{3.12-b}$$

It is eminently clear that (3.12–a) and (3.12–b) are identical, and, moreover, they represent a row–by–row detailed form of (3.6) for the discrete-time case. (ii) *continuous-time case*. The fulfillment of the flow-invariance condition stated in Definition 3.1, means, according to [10, Chapter 2, Lemma 4.2],

that, for any $t \in \mathbb{R}_+$, the inequalities given below hold:

$$a_{ii}h_i(t) + \sum_{j=1,\, j\neq i}^{n} a_{ij}x_j(t) \leq \dot{h}_i(t), \quad i = 1, \ldots, n, \tag{3.13−a}$$

$$-\dot{h}_i(t) \leq a_{ii}(-h_i(t)) + \sum_{j=1,\, j\neq i}^{n} a_{ij}x_j(t), \quad i = 1, \ldots, n, \tag{3.13−b}$$

with the same constrains on a_{ij} and $x_j(t)$ as in (3.10) and (3.11), respectively. If these constrains are adequately applied to (3.13–a) and (3.13–b), and one makes use of notation (3.8), inequalities (3.13–a) and (3.13–b) can be equivalently written as:

$$\overline{a}_{ii}h_i(t) + \sum_{j=1,\, j\neq i}^{n} \overline{a}_{ij}h_j(t) \leq \dot{h}_i(t), \quad i = 1, \ldots, n, \tag{3.14−a}$$

$$-\dot{h}_i(t) \leq \overline{a}_{ii}(-h_i(t)) + \sum_{j=1,\, j\neq i}^{n} \overline{a}_{ij}(-h_j(t)), \quad i = 1, \ldots, n. \tag{3.14−b}$$

Obviously (3.14–a) and (3.14–b) coincide, and, moreover, they represent a row–by–row detailed form of (3.6) for the continuous-time case.

It is worth mentioning that the proof for (ii) continuous-time case can also be conducted by using results presented in [15]. □

Remark 3.1. Matrix $\overline{\mathbf{A}}$ defined in Theorem 3.1 has a particular structure, namely: (i) $\overline{\mathbf{A}}$ built according to (3.7) is *nonnegative*, (ii) $\overline{\mathbf{A}}$ built according to (3.8) is *essentially nonnegative* (i.e., the off-diagonal entries are nonnegative) and it can be expressed as a sum of two real matrices:

$$\overline{\mathbf{A}} = \overline{\mathbf{A}}_D + \overline{\mathbf{A}}_E, \tag{3.15}$$

where $\overline{\mathbf{A}}_D$ is a diagonal matrix containing the diagonal entries of $\overline{\mathbf{A}}$, and $\overline{\mathbf{A}}_E \geq 0$ is a nonnegative matrix with zero diagonal entries, containing the off-diagonal entries of $\overline{\mathbf{A}}$.

Remark 3.2. In general, matrix $\overline{\mathbf{A}}$ does not belong to the interval matrix $\mathbf{A}^I$ (3.3), but $\overline{\mathbf{A}}$ dominates the interval matrix $\mathbf{A}^I$ (3.3) in the sense of the following matrix inequalities.
(i) For discrete-time case, when $\overline{\mathbf{A}}$ is built according to (3.7), for any matrix $\mathbf{A} \in \mathbf{A}^I$ (3.3), one can write:

$$\forall\, \mathbf{A} \in \mathbf{A}^I : \quad |\mathbf{A}| \leq \overline{\mathbf{A}}. \tag{3.16}$$

(ii) For continuous-time case, when $\overline{\mathbf{A}}$ is built according to (3.8), for any matrix $\mathbf{A} \in \mathbf{A}^I$ (3.3), one can write:

$$\forall\, \mathbf{A} = \mathbf{A}_D + \mathbf{A}_E \in \mathbf{A}^I : \quad \mathbf{A}_D \leq \overline{\mathbf{A}}_D, \quad |\mathbf{A}_E| \leq \overline{\mathbf{A}}_E, \tag{3.17}$$

where $\mathbf{A}_D$ and $\mathbf{A}_E$ are two real matrices containing the diagonal and off-diagonal entries of $\mathbf{A}$, respectively, and allow expressing any matrix $\mathbf{A}$ in a form similar to (3.15).

Remark 3.3. Inequalities (3.16) and (3.17) show that matrix $\overline{\mathbf{A}}$, built according to (3.7) and (3.8), respectively, dominates a *symmetric convex set* of matrices $\mathbf{M}^C$, including the interval matrix $\mathbf{A}^I$ (3.3), which is defined as follows:
(i) for the discrete-time case:

$$\mathbf{M}^C = \{\mathbf{M} \in \mathbb{R}^{n\times n} : \ |\mathbf{M}| \leq \overline{\mathbf{A}}\}; \tag{3.18}$$

(ii) for the continuous-time case:

$$\mathbf{M}^C = \{\mathbf{M} = \mathbf{M}_D + \mathbf{M}_E \in \mathbb{R}^{n\times n} :\ \mathbf{M}_D \leq \overline{\mathbf{A}}_D,\ |\mathbf{M}_E| \leq \overline{\mathbf{A}}_E\}, \tag{3.19}$$

where $\mathbf{M}_D$ and $\mathbf{M}_E$ are two real matrices containing the diagonal and off-diagonal entries of $\mathbf{M}$, respectively, and allow expressing any matrix $\mathbf{M}$ in a form similar to (3.15). In the space of coefficients, (i) for the discrete-time case, symmetry with respect to the space origin refers to all the elements, and (ii) for the continuous-time case, symmetry with respect to the space origin refers only to the extra-diagonal elements.

Based on the domination of matrix $\overline{\mathbf{A}}$ pointed out by Remark 3.2., we will also say that the linear constant system:

$$\mathbf{x}'(t) = \overline{\mathbf{A}}\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0; \quad t, t_0 \in \mathbb{T}, \quad t \geq t_0, \tag{3.20}$$

dominates any system belonging to the interval matrix system (3.2).

Remark 3.4. According to Remark 3.3, one can simply notice that linear constant system (3.20) dominates the whole family of systems generated with matrices belonging to the convex set $\mathbf{M}^C$ defined by (3.18), or (3.19), respectively. This fact should be interpreted as a key point in model construction when dealing with flow invariance, because $\overline{\mathbf{A}}$ ultimately characterizes an interval matrix system with symmetric uncertainties in the sense formulated in Remark 3.3, even if interval matrix $\mathbf{A}^I$ (3.3) does not exhibit such a symmetry.

Although system (3.20) is not necessarily a member of the family of systems that defines interval matrix system (3.2), the state-transition matrix

of constant system (3.20)

$$\overline{\Phi}(t_0,t)=\begin{cases}\overline{\mathbf{A}}^{t-t_0}; & t,t_0\in\mathbb{T}=\mathbb{Z}_+,\quad t\geq t_0,\\ e^{\overline{\mathbf{A}}(t-t_0)}; & t,t_0\in\mathbb{T}=\mathbb{R}_+,\quad t\geq t_0,\end{cases}\tag{3.21}$$

plays an important role in characterizing the flow-invariance with respect to interval matrix system (3.2), as revealed by the next result.

Theorem 3.2. *The rectangular set with arbitrary time-dependence* $\mathcal{H}(t)$ (3.4) *is flow-invariant with respect to interval matrix system* (3.2) *if and only if*:

$$\forall\, t_0,\quad t\in\mathbb{T},\quad t\geq t_0:\quad \overline{\Phi}(t_0,t)\mathbf{h}(t_0)\leq\mathbf{h}(t).\tag{3.22}$$

Proof. (*i*) *discrete-time case. Su ciency.* If (3.22) is true, then, by taking in (3.22) $t=k+1$, $t_0=k$, with arbitrary $k\in\mathbb{Z}_+$, one obtains condition (3.6) stated in Theorem 3.1. *Necessity.* Using condition (3.6), for arbitrary t, $t_0\in\mathbb{Z}_+$, with $t\geq t_0$, one can write the sequence of inequalities:

$$\overline{\mathbf{A}}^{t-t_0}\mathbf{h}(t_0)\leq\overline{\mathbf{A}}^{t-(t_0+1)}\mathbf{h}(t_0+1)\leq\ldots\leq\overline{\mathbf{A}}\mathbf{h}(t-1)\leq\mathbf{h}(t),\tag{3.23}$$

which proves that (3.22) is true.
(*ii*) *continuous-time case*. Along the lines of the proof in [13, Theorem 3], it is a simple matter to show that inequality (3.6) is equivalent to inequality (3.22). □

Remark 3.5. A direct consequence of Theorems 3.1 or 3.2 is the fact that the flow-invariance of the set $\mathcal{H}(t)$ (3.4) with respect to interval matrix system (3.2) guarantees the existence of this property for the whole *class* of rectangular sets $\tilde{\mathcal{H}}(t)$ *homothetic* to $\mathcal{H}(t)$, the time-dependence of which is defined by the vector functions:

$$\tilde{\mathbf{h}}(t)=c\,\mathbf{h}(t),\quad c>0,\quad t\in\mathbb{T}.\tag{3.24}$$

4. Flow-invariance of exponentially time-dependent rectangular sets with respect to an interval matrix system

In the current section our interest focuses on a special type of rectangular sets $\mathcal{H}(t)$ (3.4) that are flow-invariant with respect to interval matrix system (3.2), namely those sets presenting an *exponential time-dependence*, i.e.,

$$h(t)=\begin{cases}\mathbf{v}r^t,\ \mathbf{v}\in\mathbb{R}^n,\quad \mathbf{v}>0,\quad r>0,\quad t\in\mathbb{T}=\mathbb{Z}_+,\\ \mathbf{v}e^{rt},\quad \mathbf{v}\in\mathbb{R}^n,\quad \mathbf{v}>0,\quad r\in\mathbb{R},\quad t\in\mathbb{T}=\mathbb{R}_+.\end{cases}\tag{4.1}$$

Theorem 4.1. *The rectangular set with exponential time-dependence* $\mathcal{H}(t)$ (3.4) and (4.1) *is flow-invariant with respect to interval matrix system* (3.2)

if and only if the system of inequalities

$$\overline{\mathbf{A}}\mathbf{v} \leq r\mathbf{v}, \quad r \in \mathbb{R}, \quad \mathbf{v} \in \mathbb{R}^n, \quad \mathbf{v} > 0, \tag{4.2}$$

is compatible (has solutions).

Proof. The particular form of the vector function $\mathbf{h}(t)$ (4.1) is replaced in condition (3.6) stated by Theorem 3.1. □

In the light of this result it is quite natural to investigate the conditions that ensure the compatibility of system (4.2). This becomes a straightforward task by noticing that system (4.2) has the same form as system (2.6) and applying Lemma 2.2. Thus we have proved the following result:

Theorem 4.2. *The rectangular set with exponential time-dependence* $\mathcal{H}(t)$ (3.4) and (4.1) *is flow-invariant with respect to interval matrix system* (3.2) *if and only if the following condition is met:*

$$\lambda_{\max}(\overline{\mathbf{A}}) \leq r, \tag{4.3}$$

where $\lambda_{\max}(\overline{\mathbf{A}})$ *denotes the dominant eigenvalue of matrix* $\overline{\mathbf{A}}$ *in the sense formulated in Lemma* 2.1.

Remark 4.1. Theorem 4.2 points out two distinct situations in approaching the resolution of the system of inequalities (4.2), regardless of the procedure (3.7) or (3.8) used for building matrix $\overline{\mathbf{A}}$:

$$1^\circ \quad \forall r \in \mathbb{R}, \quad r \geq \lambda_{\max}(\overline{\mathbf{A}}) \qquad \exists \mathbf{v} \in \mathbb{R}^n, \quad \mathbf{v} > 0 : \quad \overline{\mathbf{A}}\mathbf{v} \leq r\mathbf{v}; \tag{4.4}$$

$$2^\circ \quad \forall \mathbf{v} \in \mathbb{R}^n, \quad \mathbf{v} > 0 \qquad \exists r \in \mathbb{R}, \quad r \geq \lambda_{\max}(\overline{\mathbf{A}}) : \quad \overline{\mathbf{A}}\mathbf{v} \leq r\mathbf{v}. \tag{4.5}$$

Remark 4.2. For arbitrary choices of both $r \geq \lambda_{\max}(\overline{\mathbf{A}})$ and $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} > 0$, the system of inequalities (4.2) may not be satisfied.

In terms of the flow-invariance of the set $\mathcal{H}(t)$ (3.4) and (4.1) with respect to interval matrix system (3.2), these two remarks on the compatibility of system (4.2), along with Theorem 4.1 reveal a meaningful correlation between the initial size of the rectangular set (specified by the vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} > 0$, in (4.1)) and the exponential time-dependence (specified by $r \in \mathbb{R}$ in (4.1)), correlation which cannot be violated.

5. Componentwise asymptotic stability of an interval matrix system

As resulted from Definition 3.1, the flow-invariance of the rectangular set $\mathcal{H}(t)$ (3.4) with respect to interval matrix system (3.2) expresses a property of the state-space trajectories, which, once initialized inside $\mathcal{H}(t)$, do not leave it any longer. In the case when the time-dependent evolution of $\mathcal{H}(t)$ approaches the state-space origin for large $t \in \mathbb{T}$, this property reflects a

special type of asymptotically stable dynamics for interval matrix system (3.2), which is rigorously defined as follows.

Definition 5.1. Interval matrix system (3.2) is called *componentwise asymptotically stable* if there exists an arbitrarily time-dependent rectangular set $\mathcal{H}(t)$ (3.4), also meeting the additional condition

$$\lim_{t\to\infty} \mathbf{h}(t) = 0, \tag{5.1}$$

such that $\mathcal{H}(t)$ is flow-invariant with respect to interval matrix system (3.2).

Remark 5.1. According to Definition 5.1, if interval matrix system (3.2) is componentwise asymptotically stable, then it is also *asymptotically stable* in the sense of the standard definition based on any consistent norm in $\mathbb{R}^n$. The converse statement is in general not true. However in the final part of this section we are going to mention some classes of interval matrix systems for which asymptotic stability is equivalent to the componentwise asymptotic stability.

Theorem 5.1. *Interval matrix system* (3.2) *is componentwise asymptotically stable if and only if* $\overline{\mathbf{A}}$ *is a stable matrix:*
(i) *in the sense of Schur for the discrete-time case,*
(ii) *in the sense of Hurwitz for the continuous-time case.*

Proof. *Sufficiency.* Let $\overline{\mathbf{A}}$ be a stable matrix and consider the vector function $\mathbf{h}(t)$ given by the state-space trajectory of system (3.20) initialized in an arbitrary $\mathbf{h}(t_0) > 0$. Hence, $\mathbf{h}(t)$ meets condition (5.1) formulated in Definition 5.1. Moreover, according to (3.20), one can write the equality:

$$\forall\, t \in \mathbb{T}: \quad \overline{\mathbf{A}}\mathbf{h}(t) = \mathbf{h}'(t), \tag{5.2}$$

which proves that condition (3.6) in Theorem 3.1 is fulfilled. Thus we have shown that interval matrix system (3.2) is componentwise asymptotically stable.

Necessity. Assume that interval matrix system (3.2) is componentwise asymptotically stable. Let $\mathcal{H}(t)$ (3.4) and (5.1) be a rectangular set that is flow-invariant with respect to interval matrix system (3.2). From (5.1) it results that there exist two time instants $t > t_0$, t, $t_0 \in \mathbb{T}$, such that

$$\mathbf{h}(t) \leq \phi\mathbf{h}(t_0), \quad \text{with} \quad 0 < \phi < 1. \tag{5.3}$$

Now, combining condition (3.22) (with (3.21) from Theorem 3.2) with inequality (5.3), one can notice that:

$$\overline{\Phi}(t_0, t)\mathbf{h}(t_0) \leq \phi\mathbf{h}(t_0), \quad \text{with} \quad 0 < \phi < 1. \tag{5.4}$$

In (5.4), $\overline{\Phi}(t_0, t)$ is a nonnegative matrix; this fact results directly from (3.21) for (i) the discrete-time case, and it is proved in [13] for (ii) the continuous-time case. Hence, system (5.4) has the same form as system of inequalities (2.6) and, according to Lemma 2.2, it is compatible if and only if

$$\lambda_{\max}(\overline{\Phi}(t_0, t)) \leq \phi < 1. \tag{5.5}$$

Relying on the links between the eigenvalues of $\overline{\mathbf{A}}$ and the eigenvalues of $\overline{\Phi}(t_0, t)$, one can immediately see that
(i) for discrete-time case, $|\lambda_i(\overline{\mathbf{A}})| < 1$, $i = 1, \ldots, n$, i.e., $\overline{\mathbf{A}}$ is Schur stable,
(ii) for continuous-time case, $\Re e[\lambda_i(\overline{\mathbf{A}})] < 0$, $i = 1, \ldots, n$, i.e., $\overline{\mathbf{A}}$ is Hurwitz stable. □

Based on the definitions of componentwise asymptotic stability and standard asymptotic stability of interval matrix system (3.2), Remark 5.1 shows that the former involves the latter; this statement exploits the geometric properties of the interval matrix system state-space trajectories. Now, once we have got the necessary and sufficient condition expressed by Theorem 5.1, we are interested to explore the algebraic background of Remark 5.1 in terms of *matrix stability*. For the proposed development, the result formulated below plays a crucial role.

Theorem 5.2. *Let $\overline{\mathbf{A}}$ be a matrix built according to* (3.7) *or* (3.8) *from the interval matrix $\mathbf{A}^I$* (3.3). *The eigenvalue $\lambda_{\max}(\overline{\mathbf{A}})$, which dominates the spectrum of $\overline{\mathbf{A}}$ in the sense formulated by Lemma* 2.1, *also dominates the spectrum of any arbitrary matrix $\mathbf{A} \in \mathbf{A}^I$ in the same sense, as follows:*
(i) *for $\overline{\mathbf{A}}$ built according to* (3.7)

$$\forall\, \mathbf{A} \in \mathbf{A}^I : \quad |\lambda_i(\mathbf{A})| \leq \lambda_{\max}(\overline{\mathbf{A}}), \quad i = 1, \ldots, n; \tag{5.6}$$

(ii) *for $\overline{\mathbf{A}}$ built according to* (3.8)

$$\forall\, \mathbf{A} \in \mathbf{A}^I : \quad \Re e[\lambda_i(\overline{\mathbf{A}})] \leq \lambda_{\max}(\overline{\mathbf{A}}), \quad i = 1, \ldots, n. \tag{5.7}$$

Proof. (i) for $\overline{\mathbf{A}}$ built according to (3.7), the matrix inequality $|\mathbf{A}| \leq \overline{\mathbf{A}}$ holds for any $\mathbf{A} \in \mathbf{A}^I$ and therefore $|\lambda_i(\mathbf{A})|$, $i = 1, \ldots, n$, cannot exceed the spectral radius of $\overline{\mathbf{A}}$, i.e. $\lambda_{\max}(\overline{\mathbf{A}})$ (cf. [2]).
(ii) for $\overline{\mathbf{A}}$ built according to (3.8), let us consider $s > 0$, such that

$$a_{ii}^- + s \geq 0, \quad i = 1, \ldots, n. \tag{5.8}$$

Thus, for any $\mathbf{A} \in \mathbf{A}^I$, one can write the matrix inequality

$$|\mathbf{A} + s\mathbf{I}| \leq \overline{\mathbf{A}} + s\mathbf{I}, \tag{5.9}$$

which shows that $|\lambda_i(\mathbf{A}+s\mathbf{I})|$, $i = 1, \ldots, n$, cannot exceed the spectral radius of $\mathbf{A} + s\mathbf{I}$ (cf. [2]). Consequently, for any $\mathbf{A} \in \mathbf{A}^I$ one gets the following

inequalities:

$$\begin{aligned}\Re e[\lambda_i(\mathbf{A})] + s = \Re e[\lambda_i(\mathbf{A} + s\mathbf{I})] \leq |\lambda_i(\mathbf{A} + s\mathbf{I})| \leq \rho(\overline{\mathbf{A}} + s\mathbf{I}) \\ = \lambda_{\max}(\overline{\mathbf{A}}) + s, \quad i = 1, \ldots, n.\end{aligned} \tag{5.10}$$

which complete the proof. □

Remark 5.2. According to Remark 3.3 the domination of the eigenvalue $\lambda_{\max}(\overline{\mathbf{A}})$ remains valid in the sense formulated in Theorem 5.2 for the whole symmetric convex set of matrices $\mathbf{M}^C$ defined by inequalities (3.18) and (3.19), respectively.

Now, resuming our discussion on the stability of matrix $\overline{\mathbf{A}}$ versus the stability of the interval matrix $\mathbf{A}^I$, we can simply formulate an immediate consequence of Theorem 5.2.

Corollary 5.1. *Let $\overline{\mathbf{A}}$ be a matrix built according to* (3.7) *or* (3.8) *from the interval matrix $\mathbf{A}^I$* (3.3)*. If $\overline{\mathbf{A}}$ is stable then $\mathbf{A}^I$ is stable:*
(*i*) *in the sense of Schur, for $\overline{\mathbf{A}}$ built according to* (3.7),
(*ii*) *in the sense of Hurwitz, for $\overline{\mathbf{A}}$ built according to* (3.8).

In this way, using only concepts specific to linear algebra, we have proved our previous statement (Remark 5.1) referring to the relationship existing for interval matrix system (3.2) between the componentwise asymptotic stability (i.e., $\overline{\mathbf{A}}$ stable) and standard asymptotic stability (i.e., $\mathbf{A}^I$ stable).

Remark 5.3. The result formulated in Corollary 5.1 can be found in [1], [11] for Schur stability of an interval matrix and in [11] for Hurwitz stability. The noticeable achievement of our work consists in revealing the *complete meaning* of the algebraic condition of stability (Schur or Hurwitz) for matrix $\overline{\mathbf{A}}$, based on the stronger concept of componentwise asymptotic stability. Thus, the stability of matrix $\overline{\mathbf{A}}$ operates only as a *su cient criterion* for the asymptotic stability of interval matrix system (3.2) – result already known, whereas it represents a *necessary* and *su cient condition* for the componentwise asymptotic stability of interval matrix system (3.2) – new result. Moreover, for $\overline{\mathbf{A}}$ stable, our recent insight demonstrates that the usage of the dominant eigenvalue $\lambda_{\max}(\overline{\mathbf{A}})$ as a stability margin for interval matrix system (3.2) (proposed in [4] or [1] for standard asymptotic stability) fully characterizes only the situation when the margin refers to the componentwise asymptotic stability of interval matrix system (3.2).

In the light of these comments, if interval matrix system (3.2) is componentwise asymptotically stable, the dominant eigenvalue $\lambda_{\max}(\overline{\mathbf{A}})$ is expected to provide supplementary information with regard to the time-dependent evolution of those sets $\mathcal{H}(t)$ (3.4) and (5.1) that are flow-invariant with respect to interval matrix system (3.2).

Theorem 5.3. *Let interval matrix system* (3.2) *be componentwise asymptotically stable and consider a set* $\mathcal{H}(t)$ (3.4) and (5.1) *that is flow-invariant with respect to interval matrix system* (3.2)*. For any two arbitrary time instants* t_0, $t \in \mathbb{T}$, $t > t_0$ *, for which* $\mathbf{h}(t) < \mathbf{h}(t_0)$*, define the decreasing rate of* $\mathcal{H}(t)$ *between* t_0 *and* t*, denoted by* $d(t_0, t)$*, as the minimal subunitary value ensuring the fulfillment of the following inequality*:

$$\mathbf{h}(t) \leq d(t_0, t)\mathbf{h}(t_0). \tag{5.11}$$

The decreasing rate of $\mathcal{H}(t)$ *has a lower bound that can be expressed as a time-dependent exponential function of parameter* $\lambda_{\max}(\overline{\mathbf{A}})$*:*
(i) *for discrete-time case:* $(\lambda_{\max}(\overline{\mathbf{A}}))^{t-t_0} \leq d(t_0, t)$,
(ii) *for continuous-time case:* $e^{\lambda_{\max}(\overline{\mathbf{A}})(t-t_0)} \leq d(t_0, t)$.

Proof. Combining (5.11) with the flow-invariance condition (3.22) given in Theorem 3.2, we get:

$$\overline{\Phi}(t_0, t)\mathbf{h}(t_0) \leq d(t_0, t)\mathbf{h}(t_0), \tag{5.12}$$

which is a system of inequalities identical to (5.4); hence its compatibility is ensured by condition (5.5) rewritten for $\phi = d(t, t_0)$, i.e.,

$$\lambda_{\max}(\overline{\Phi}(t_0, t)) \leq d(t_0, t) < 1. \tag{5.13}$$

The proof is complete if we make use of the well-known link existing between $\lambda_{\max}(\overline{\mathbf{A}})$ and $\lambda_{\max}(\overline{\Phi}(t_0, t))$ for both discrete-time and continuous-time cases. □

Although standard asymptotic stability of interval matrix system (3.2) represents only a necessary condition for the componentwise asymptotic stability, one can identify several classes of interval matrix systems for which this condition is also sufficient. Obviously such situations require a deep insight into the geometrical properties of state-space trajectories that will be addressed as a separate study. In this section we restraint our presentation to three classes of interval matrix systems, whose particular structures correspond to special types of interval matrices $\mathbf{A}^I$ (3.3), where the stability of $\mathbf{A}^I$ is equivalent to the stability of $\overline{\mathbf{A}}$.

Remark 5.4. (i) For the discrete-time case, assume that all the entries of $\mathbf{A}^I$ (3.3) are either nonnegative, or nonpositive. Interval matrix system (3.2) is componentwise asymptotically stable if and only if it is asymptotically stable.

(ii) For the continuous-time case, assume that all the extra-diagonal entries of $\mathbf{A}^I$ (3.3) are nonnegative. Interval matrix system (3.2) is componentwise asymptotically stable if and only if it is asymptotically stable.

Remark 5.5. For both discrete-time and continuous-time case, assume that $\mathbf{A}^I$ (3.3) is either lower- or upper-triangular. Interval matrix system (3.2) is componentwise asymptotically stable if and only if it is asymptotically stable.

Remark 5.6. (*i*) For the discrete-time case, assume that $\mathbf{V}$ is an extreme vertex of the hyperrectangle described in $\mathbb{R}^{n\times n}$ by $\mathbf{A}^I$ (3.3), such that $|\mathbf{V}| = \overline{\mathbf{A}}$, i.e., $\mathbf{V}$ is also a vertex of the convex set of matrices $\mathbf{M}^C$ defined by (3.18). Assume that $\mathbf{V}$ or $-\mathbf{V}$ is a Morishima matrix. Interval matrix system (3.2) is componentwise asymptotically stable if and only if it is asymptotically stable, according to [11].

(*ii*) For the continuous-time case, assume that $\mathbf{V} = \mathbf{V}_D + \mathbf{V}_E$ is an extreme vertex of the hyperrectangle described in $\mathbb{R}^{n\times n}$ by $\mathbf{A}^I$ (3.3), such that $\mathbf{V}_D = \overline{\mathbf{A}}_D$, $|\mathbf{V}_E| = \overline{\mathbf{A}}_E$, with $\overline{\mathbf{A}}_D$, $\overline{\mathbf{A}}_E$ defined by (3.15), i.e., $\mathbf{V}$ is also a vertex of the convex set of matrices $\mathbf{M}^C$ defined by (3.19). Assume that $\mathbf{V}_E$ or $-\mathbf{V}_E$ is a Morishima matrix. Interval matrix system (3.2) is componentwise asymptotically stable if and only if it is asymptotically stable, according to [11].

6. Componentwise exponential asymptotic stability of an interval matrix system

In Section 4 we dealt with the flow-invariance of rectangular sets presenting an exponentially time-dependent evolution, described by the vector function $\mathbf{h}(t)$ (4.1). If in (4.1) we consider just decaying exponential (i.e. $r < 1$ for discrete-time case, and $r < 0$ for continuous-time case), $\mathbf{h}(t)$ will also satisfy condition (5.1) requested by the definition of componentwise asymptotic stability. This fact motivates us to give a new definition.

Definition 6.1. Interval matrix system (3.2) is called *componentwise exponentially asymptotically stable* if there exists an exponentially time-dependent rectangular set $\mathcal{H}(t)$ (3.4) and (4.1), also meeting the additional condition (5.1), such that $\mathcal{H}(t)$ is flow-invariant with respect to interval matrix system (3.2).

At this point, since Definition 6.1 is more restrictive than Definition 5.1 with regard to the time-dependence of the rectangular set $\mathcal{H}(t)$, it is perfectly reasonable to explore the relationship between the componentwise asymptotic stability and componentwise exponential asymptotic stability of interval matrix system (3.2).

Theorem 6.1. *Interval matrix system* (3.2) *is componentwise asymptotically stable if and only if it is componentwise exponential asymptotically stable.*

Proof. *Sufficiency.* It is obvious according to Definition 6.1.

Necessity. If interval matrix system (3.2) is componentwise asymptotically stable, then Theorem 5.1 guarantees that $\overline{\mathbf{A}}$ is stable. This means that we can find $r \in \mathbb{R}$ such that
(i) for discrete-time case $\lambda_{\max}(\overline{\mathbf{A}}) \leq r < 1$,
(ii) for continuous-time case $\lambda_{\max}(\overline{\mathbf{A}}) \leq r < 0$.

Thus, according to Theorem 4.2, we prove the existence of $\mathbf{h}(t)$ (4.1) with decaying exponential, i.e., interval matrix system (3.2) is componentwise exponential asymptotically stable. □

The equivalence between the componentwise exponential asymptotic stability and the componentwise asymptotic stability stated in Theorem 6.1 is not a surprising result, if we take into consideration the linearity of the dynamics exhibited by interval matrix system (3.2). For nonlinear dynamics this result might not be preserved as suggested in [15].

For testing the componentwise (exponential) asymptotic stability of interval matrix system (3.2), besides the investigation of Schur or Hurwitz stability for matrix $\overline{\mathbf{A}}$, one can apply the following criteria, which are equivalent to the stability of $\overline{\mathbf{A}}$, according to some results presented in [2].

Theorem 6.2. *Interval matrix system* (3.2) *is componentwise exponential asymptotically stable if and only if:*
(i) *for the discrete-time case,* $\mathbf{I} - \overline{\mathbf{A}}$ *is an M-matrix,*
(ii) *for the continuous-time case,* $-\overline{\mathbf{A}}$ *is an M-matrix.*

Theorem 6.3. *Interval matrix system* (3.2) *is componentwise exponential asymptotically stable if and only if there exists a set of generalized Gershgorin's disks associated with matrix* $\overline{\mathbf{A}}$*, lying in a prescribed region of the complex plane* s*, namely:*
(i) $|s| < 1$*, for the discrete-time case,*
(ii) $\Re e[s] < 0$*, for the continuous-time case.*

Theorem 6.4. *Interval matrix system* (3.2) *is componentwise exponential asymptotically stable if and only if the leading principal minors,* Δ_k*,* $k = 1, 2, \ldots, n$*, built with the entries of the matrix:*
(i) $\overline{\mathbf{A}} - I$*, for the discrete-time case,*
(ii) $\overline{\mathbf{A}}$*, for the continuous-time case,*
meet the conditions:

$$(-1)^k \Delta_k > 0, \quad k = 1, 2, \ldots, n. \tag{6.1}$$

Given interval matrix system (3.2), once the stability of matrix $\overline{\mathbf{A}}$ is proved, the practical search for those sets $\mathcal{H}(t)$, (3.4), (4.1), and (5.1) that are flow-invariant with respect to interval matrix system (3.2) should rely

on solving system (4.2) along the lines of (4.4) or (4.5) (Remark 4.1), taking an adequate upper bound for r. In this context, it is worth underlining the usefulness of Remark 4.2, in order to avoid improper searches.

7. Conclusions

Given interval matrix system (3.2) with discrete-time or continuous-time dynamics, we have first defined (Definition 3.1) the flow-invariance with respect to this interval matrix system of the rectangular sets $\mathcal{H}(t)$ (3.4) with arbitrary time-dependence. Then, we have derived two characterizations (Theorems 3.1 and 3.2) of flow-invariance emphasizing the role played by the constant matrix $\overline{\mathbf{A}}$ (3.7), or (3.8), which, in a certain sense (expressed by matrix inequalities), dominates the interval matrix $\mathbf{A}^I$ (3.3). For an exponential time-dependence (4.1) of the rectangular set $\mathcal{H}(t)$, the flow-invariance property is equivalent to the compatibility of the system of inequalities (4.2) (Theorem 4.1), which is further developed as a necessary and sufficient condition referring to the location of the dominant eigenvalue of matrix $\overline{\mathbf{A}}$ (Theorem 4.2).

The flow-invariance concept allows defining the componentwise asymptotic stability of interval matrix system (3.2) (Definition 5.1) and proving its direct and converse link with the stability of matrix $\overline{\mathbf{A}}$ (Theorem 5.1). A refined algebraic approach shows that the dominant eigenvalue of $\overline{\mathbf{A}}$ is also dominant for the eigenvalues of any matrix belonging to the interval matrix which defines interval matrix system (3.2) (Theorem 5.2). The stability condition for matrix $\overline{\mathbf{A}}$ has already been used by other authors as a sufficient criterion for the standard asymptotic stability of interval matrix systems (defined in terms of any consistent norm), but only now is revealed its complete meaning (including the necessity part too), as reflecting the supplementary requirements of the stronger concept of componentwise asymptotic stability. A brief discussion focuses on three particular classes of interval matrix systems, which are componentwise asymptotically stable if and only if they are asymptotically stable. For a componentwise asymptotically stable interval matrix system (3.2), it is shown that the decreasing rate of any set $\mathcal{H}(t)$ (3.4) which is flow-invariant with respect to interval matrix system (3.2) cannot exceed an exponential function parameterized by the dominant eigenvalue of $\overline{\mathbf{A}}$ (Theorem 5.3).

Finally, we define the componentwise exponential asymptotic stability of interval matrix system (3.2) (Definition 6.1) and prove that it is equivalent to the componentwise asymptotic stability (Theorem 6.1), as expected due to the linearity exhibited by the dynamics of interval matrix system (3.2). For testing the componentwise (exponential) asymptotic stability of interval

matrix system (3.2), some criteria, equivalent to the stability of matrix $\overline{\mathbf{A}}$, are also given (Theorem 6.2, 6.3 and 6.4). Part of the results presented in this paper represent a generalization for interval matrix systems of the previous work developed by one of the authors for continuous-time linear constant systems. Future research should provide a deeper insight into the geometric aspects of the flow-invariance property with respect to interval matrix system (3.2), placing emphasis on the role played by the system eigenstructure.

References

[1] P. H. Bauer and K. Premaratne, *Time-invariant versus time-variant stability of interval matrix systems*, in M. Jamshidi, M. Mansour and B.D.O. Anderson (Eds.) "Fundamentals of Discrete-Time Systems: A Tribute to Professor E. I. Jury", TSI Press, Albuquerque, 1993, 181–188.

[2] A. Berman and R. J. Plemmons, "Nonnegative Matrices in the Mathematical Sciences", Academic Press, New York, 1979.

[3] H. Brezis, *On a characterization of flow-invariant sets*, Comm. Pure Appl. Math., 23 (1970), 261–263.

[4] J. Chen, *Su cient condition on stability of interval matrices: connections and new results*, IEEE Trans. on Aut. Control, 37 (1992), 541–544.

[5] M. G. Crandall, *A generalization of Peano's existence theorem and flow-invariance*, Proc. Amer. Math. Soc., 36 (1972), 151–155.

[6] M. Hukuhara, *Sur la theorie des equations di erentieles ordinaires*, J. Fac. Sci. Univ. Tokio, Math., 7 (1954), 483-510.

[7] R. H. Martin jr., *Di erential equations on closed subsets of a Banach space*, Trans. Amer. Math. Soc., 179 (1973), 399–414.

[8] D. Motreanu and H.N. Pavel, "Tangency, flow invariance for differential equations and optimization problems", Marcel Dekker, New York, 1999.

[9] M. Nagumo, *Über die Lage der Integralkurven gewöhnlicher Di erentialgleichungen*, Proc. Phys. Math. Soc., Japan, 24 (1942), 551–559.

[10] H. N. Pavel, "Differential Equations Flow-Invariance and Applications", Pitman, Boston, 1984.

[11] M. E. Sezer and D. D. Šiljak, *On stability of interval matrices*, IEEE Trans. on Aut. Control, 39 (1994), 368–371.

[12] J. Stoer and C. Witzgall, *Transformations by diagonal matrices in a normed space*, Numerische Math, 4 (1962), 158–171.

[13] M. Voicu, *Free response characterization via flow-invariance*, in "Preprints of the 9-th World Congress of IFAC", Vol. 5, 12–17, Budapest, 1984.

[14] M. Voicu, *Componentwise asymptotic stability of linear constant dynamical systems*, IEEE Trans. on Aut. Control, 10 (1984), 937–939.

[15] M. Voicu, *On the application of the flow-invariance method in control theory an design*, in "Preprints of the 10-th World Congress of IFAC", Vol. 8, 364–369, Munich, 1987.